Novel phases of planar fermionic systems

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Abstract. We discuss a *family* of planar (two-dimensional) systems with the following phase structure: a Fermi liquid, which goes via a second-order transition (with a nonclassical exponent even in the mean field) to an intermediate, inhomogeneous state (with nonstandard ordering momentum), which in turn goes via a first-order transition to a state with a canonical order parameter. We analyse two examples: (i) a superconductor in a parallel magnetic field for which the inhomogeneous state is obtained for $1.86T_c \lesssim B \lesssim 1.86\sqrt{2}T_c$ where T_c is the critical temperature (in kelvin) of the superconductor without a field and B is measured in tesla; and (ii) spinless (or spin-polarized) fermions near half-filling where a similar, sizeable window (which grows in size with anisotropy) exists for the intermediate CDW phase at a momentum different from (π, π). We discuss the experimental conditions for realizing and observing these phases and the renormalization group approach to the transitions.

1. Introduction

Zero-temperature quantum critical phenomena have been a subject of intense theoretical and experimental investigation in recent years. Of particular interest to us has been the renormalization group (RG) description of the transitions. In our pursuit of the RG description of two-dimensional fermionic systems, we came across a *family* of systems which exhibit similar phase structure and similar phase transitions separating the phases in mean-field theory. Since this phase structure seems generic and experimentally accessible today, we have chosen to make it the focus of this paper, only briefly discussing the RG programme that led us to it in the first place.

The systems in question exhibit three phases as a control parameter I is varied. Since the physics depends on the ratio W/I, where W is the interaction strength, we can either vary I at fixed W (which is what happens in experiments), or vary W at fixed I in a theoretical discussion. The phases of the system are a weak-coupling (large-I) Fermi liquid phase, and a strong-coupling phase with conventional order, separated by an intermediate, inhomogeneous phase with nonstandard ordering momentum. The intermediate phase, which is the most interesting, is *accessible over a broad window in I* (in clean systems) thanks to two dimensionality. Indeed, in anisotropic systems that are more one dimensional, the window is broader.

We will illustrate our theme using two concrete examples: a system of electrons with an attractive coupling between up and down spins and a system of spinless fermions with nearest-neighbour repulsion. (We hasten to add that we will indicate how these 'spinless

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fermions' are to be realized experimentally by partially polarizing ordinary electrons in a manner to be specified.)

Consider first the system of free electrons—by which we always mean Landau's quasiparticles—with a circular Fermi surface (FS). They are unstable to the BCS state at arbitrarily small attraction W. The instability may be traced back to the time-reversal symmetry of the spectrum and hence of the FS: $E(K) = E(-K)$. Now, this instability has a nice RG description. It was shown in [1] that in an RG scheme in which modes around the Fermi surface are symmetrically eliminated in thin slices, the Fermi liquid appears as a fixed point and the above-mentioned interaction appears as a marginally relevant term. The existence of a steady flow of the coupling in the simple mode-elimination scheme was also seen to result from the special symmetry of the Fermi surface.

Consider next a system of free spinless fermions, at half-filling, with nearest-neighbour hopping on a rectangular lattice. This system is unstable to arbitrarily weak nearest-neighbour repulsion W which drives it to a charge-density-wave (CDW) state at momentum $Q = (\pi, \pi)$. The immediate instability is again due to a special symmetry of the FS: due to particle–hole symmetry and nesting, if K lies on the Fermi surface, so does $K + Q$. This instability also has an RG description, and, as shown in [1], the flow again occurs thanks to the symmetry of the spectrum which changes sign under $K \to K + Q$.

We started by asking if such an RG description could be extended to transitions at nonzero coupling. A necessary prelude to such an enterprise was finding systems which exhibited such transitions. It was at that stage that we ran into the systems discussed here.

Consider first the superconductor. To move the transition to a nonzero, but small coupling (this smallness will allow a perturbative RG analysis) we must destroy the time-reversal symmetry of its FS by a small amount. To this end, let us apply a magnetic field B *parallel* to the plane. This parallel field has no effect on the orbital motion but causes a Zeeman splitting of size $2\mu B \equiv 2I$, where μ is the electron's magnetic moment. We expect that at very small fields the superconductor will be stable, while for very large fields the Fermi liquid will be stable.

Our strategy for showing that there exists an intermediate inhomogeneous phase is as follows. We start at the BCS end and ask when the BCS state yields to the Fermi liquid as the field is *raised*. Then we go the Fermi liquid end at large I and ask what happens to it as the field is *lowered*. We find that it first becomes unstable to inhomogeneous pairing *before* the BCS state 'beats' it in energy. There is clearly a window where neither the BCS state or the Fermi liquid can be the ground state.

Here are some details. Consider the response of the BCS paired state to a small magnetic field B. Since the spins form a singlet they ignore the field altogether. In the meantime, as B increases, the Fermi liquid energy decreases because it is getting polarized. Beyond the Chandrasekhar–Clogston [2] limit $I_F = \Delta_0/\sqrt{2}$ (where Δ_0 is the BCS order parameter in the absence of the field), the Fermi liquid has lower energy than the BCS state and one may expect a first-order transition (hence subscript F on I_F) to the polarized Fermi liquid (PFL). But what happens is that by this point a state with inhomogeneous order has 'beaten' the Fermi liquid, and the first-order transition is really to this state.

To see all this, let us go to the weak-coupling end (large I) where it is natural to first find the FS in the presence of the field and *then* turn on the attraction. The radii of the up and down FS now differ by $q = 2I/v_F$, where v_F is the Fermi velocity, *hereafter set equal to unity*. The PFL is stable to pairing fluctuations, at least for small coupling or large I. If we perform the sum over Cooper bubbles and evaluate the pairing susceptibility *as I is lowered*, we find two interesting features:

(i) the PFL susceptibility first diverges at $I_q = \Delta_0$, *which is $\sqrt{2}$ times bigger than the I_F at which the BCS energy rises above that of the PFL;*

(ii) this pairing is at momentum $q = 2I$.

Thus we have a window

$$\Delta_0/\sqrt{2} = I_F < I < \Delta_0 = I_q$$

when the BCS state has a higher energy than the PFL, which itself is unstable to the inhomogeneous state. To repeat, as I is increased, the BCS state will yield, via a first-order transition, not to the PFL, but to an *inhomogeneous state* with wave number $q \simeq 2I$. The phase diagram is shown in figure 1(a). Actually the inhomogeneous state will prevail below the lower limit of the above window, since at the first-order transition the BCS state has to 'beat', not the PFL, but the inhomogeneous state which has a lower energy. (For completeness we mention that at a still lower field, $I_0 = \Delta_0/2$, the $q = 0$ susceptibility of the Fermi liquid diverges. This is, however, not very significant since the corresponding instability is preempted by the other transitions.)
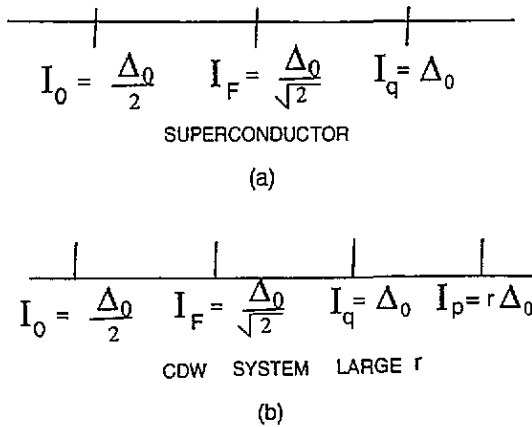

$$I_0 = \frac{\Delta_0}{2} \qquad I_F = \frac{\Delta_0}{\sqrt{2}} \qquad I_q = \Delta_0$$

SUPERCONDUCTOR

(a)

$$I_0 = \frac{\Delta_0}{2} \qquad I_F = \frac{\Delta_0}{\sqrt{2}} \qquad I_q = \Delta_0 \quad I_p = r\Delta_0$$

CDW   SYSTEM   LARGE r

(b)

Figure 1. A linear phase diagram for (a) the superconductor and (b) CDW systems as I is raised. In both cases, I_0 is the value at which the Fermi liquid susceptibility for the uniform condensate diverges. It does not correspond to a phase transition. In the CDW case I_q is the value at which the Fermi liquid susceptibility diverges for momentum in the x direction. This too is not a real transition since the system has already condensed at I_p to a state with momentum in the y direction. All momenta are measured from (π, π) in the CDW case.

Now for the spinless fermions. Here we destroy the perfect nesting of the FS by introducing a chemical potential I. (This possibility was raised in [1] along with another option: adding a second-neighbour hopping term, which we will consider only briefly.) We establish the window as before. Let us start at strong coupling (small I), and consider a CDW state at half-filling, with an order parameter Δ_0. Now imagine modifying the chemical potential. *The system does not respond to a change in the chemical potential which is moving in the gap—exactly as the superconductor did not respond to the applied field.* Once again a mean-field calculation shows that the (π, π) CDW and the FL have equal energies at $I_F = \Delta_0/\sqrt{2}$, where Δ_0 is the condensate without the chemical potential term. At the weak-coupling end, the FS has shrunk, as shown in figure 2, and becomes stable against particle–hole pairs condensing at $Q = (\pi, \pi)$ or any other momentum. An RPA calculation shows that as I is lowered, the FL becomes unstable to a CDW at a momentum different

from (π, π) at $I_p \approx r\Delta_0$, where $r > 1$ is an anisotropy parameter which measures the ratio of the hopping in the y and x directions. *Thus we have a window $\Delta_0/\sqrt{2} < I < r\Delta_0$ where an inhomogeneous state must prevail. This state is inhomogeneous in the sense that it is characterized by a momentum different from the canonical value, which in this case happens to be (π, π).* The phase diagram is shown in figure 1(b). (The significance of the points I_q, and I_0, which do not correspond to transitions, will be explained in section 3.)
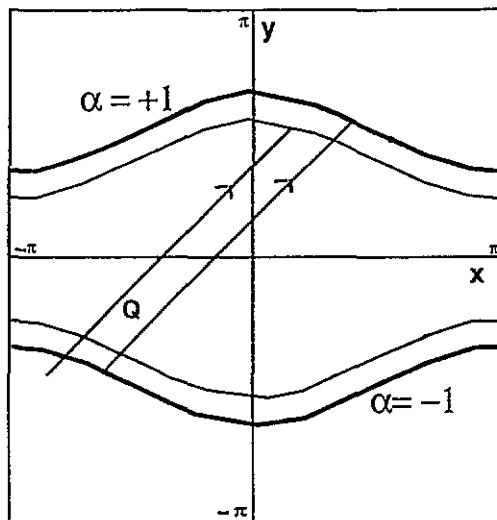

Figure 2. The dark lines show the anisotropic, two-branched, nested Fermi surface at half-filling. The vector Q connects any point on it to any other point. When the chemical potential is turned on, the surface shrinks to the thin line. There is no global nesting vector.

In both the BCS and CDW cases the mean-field second-order transition from the FL to the inhomogeneous state was found to have nonclassical exponents. For example $\beta = 2$ for the superconductor. This nonclassical answer in a mean-field calculation may be ascribed to the fact that the mode that becomes unstable has a momentum which is a singular point of the susceptibility. This singularity and the existence of a fairly large window for the inhomogeneous phase both result from the two dimensionality of the system.

The plan of the paper is as follows. In the next section we will discuss the superconductor and relate it to earlier work of Larkin and Ovchinnikov, Fulde and Ferrel [3], and Bulaevskii [5]. *In section 3 we will perform a similar analysis of the spinless-fermion system and describe how it may be experimentally realized.* Section 4 is devoted to a status report on the RG programme. We end with conclusions and avenues for future research in section 5.

2. The planar superconductor

We will begin with noninteracting electrons lying within an annulus of thickness 2Λ centred around the Fermi circle of radius K_F. We will construct a mean-field Hamiltonian that pairs the up and down electrons whose momenta lie within this band.

The energy of an up electron is

$$E_+(K) = I + \left(\frac{K^2 - K_F^2}{2m}\right) = I + v_F k = I + k \tag{1}$$

where the Fermi velocity v_F has been set equal to unity, and

$$k = K - K_F. \tag{2}$$

The energy of its partner, a down-spin electron of momentum $-K + q$, is

$$E_-(-K + q) = -I + k - q \cos \theta \tag{3}$$

where θ is the angle between K and q.

(It is understood that $-K + q$ is also constrained to lie in the same annulus. This means that certain electrons will not have partners for pairing in a state of momentum q. They must be handled correctly.)

A useful combination is

$$\frac{E_+(K) - E_-(-K + q)}{2} = I + (q/2) \cos \theta \equiv z(\theta). \tag{4}$$

Let us begin with the mean-field Hamiltonian

$$H = \frac{\Delta \Delta^*}{W} + \int_{-\Lambda}^{\Lambda} dk \int_0^{2\pi} d\theta \left[c_+^\dagger(K) c_+(K)(I + k) + c_-^\dagger(K) c_-(K)(-I + k) \right.$$
$$- \Theta(\Lambda - ||K - q| - K_F|) \left[\Delta c_+^\dagger(-K + q) c_-^\dagger(K) \right.$$
$$\left. \left. + \Delta^* c_-(K) c_+(-K + q) \right] \right] \tag{5}$$

where the condensate was taken to have the form $\Delta(x) = \Delta \exp[iq \cdot r]$, the step function $\Theta(\Lambda - ||K - q| - K_F|)$ ensures that the other partner in the pair also lies in the annulus, and the subscripts refer to the spin. The reader should not worry about factors of 2π since all key results will be given in term of observables.

By going to Bogoliubov operators $d_\pm$ and performing the usual transformation that 'kills the bad terms', we obtain

$$H = \frac{\Delta \Delta^*}{W} + \int_{-\Lambda}^{\Lambda} dk \int_0^{2\pi} d\theta \, \Theta(\Lambda - ||K - q| - K_F|)$$
$$\times \left[k - I - q \cos \theta + d_+^\dagger d_+ \left[I + (q/2) \cos \theta + \sqrt{\Delta^2 + (k - (q/2) \cos \theta)^2} \right] \right]$$
$$+ (d_-^\dagger d_- - 1) \left[-I - (q/2) \cos \theta + \sqrt{\Delta^2 + (k - (q/2) \cos \theta)^2} \right]$$
$$+ \frac{\pi q^2}{2} - 4\Lambda q. \tag{6}$$

The last two terms come from the unpaired electrons. The ground state is found by filling all negative energy states. (If we set $\Delta = 0$ above, the ground-state energy should reduce to $-2\pi(\Lambda^2 + I^2)$, the energy of the PFL.) If we subtract off the energy of the PFL, we find the relative energy

$$E(\Delta) - E(0) = 2\pi I^2 + \frac{\Delta^2}{W} + \pi \left(\frac{q^2}{4} - \Delta^2 \right) - 2\pi \Delta^2 \ln \frac{2\Lambda}{\Delta}$$
$$+ \int_0^{2\pi} d\theta \left[\theta(z - \Delta) \left[-z\sqrt{z^2 - \Delta^2} + \Delta^2 \ln \frac{z + \sqrt{z^2 - \Delta^2}}{\Delta} \right] + z \rightarrow -z \right] \tag{7a}$$

$$z(\theta) = I + \frac{q}{2} \cos \theta \tag{7b}$$

where Δ and Δ^2 stand for $|\Delta|$ and $|\Delta|^2$.

Most of our results are based on an analysis of this equation.

The first term comes from subtracting off the polarization energy of the PFL. As for the θ integral let us note that if $\Delta > I + q/2$, it does not contribute. It is clear that nonzero q is a liability in this region and the minimum energy is readily found to occur at Δ_0 given by

$$\frac{1}{W} = 2\pi \ln \frac{2\Delta}{\Delta_0} \qquad (8)$$

exactly as if B were never turned on. This is just the manifestation of the fact that the BCS system ignores the field. The condensation energy (relative to the PFL energy) is found to be

$$E(\Delta) - E(0) = \pi(2I^2 - \Delta_0^2) \qquad (9)$$

which leads to the result that $I_F = \Delta_0/\sqrt{2}$ is the field at which the BCS system yields to the Fermi liquid. (Since $\Delta_0 = \sqrt{2}I_F > I_F$ the answer is consistent with the assumptions made in deriving it.)

Now for the weak-coupling (or large-I) Fermi liquid end. Either by summing all the Cooper bubbles, or by taking the Δ^2 derivative of the energy at $\Delta = 0$, we find the formulas for χ^{-1}. (In taking this derivative it is important to note that if $I > q/2$, $z(\theta)$ is always positive and only the first term in the θ integral contributes, whereas if $q/2 > I$, the second contributes for $\Delta = 0$.) The result is

$$\chi^{-1}(q/2) = \frac{1}{W} - \int_0^{2\pi} d\theta \ \ln \frac{\Lambda}{|z|} \qquad \cdots \qquad (10)$$

$$= \frac{1}{W} - 2\pi \ln \frac{2\Lambda}{I + \sqrt{I^2 - (q/2)^2}} \qquad q/2 < I \qquad (11)$$

$$= \frac{1}{W} - 2\pi \ln \frac{2\Lambda}{q/2} \qquad q/2 > I. \qquad (12)$$

Note that at each value of I (the applied field), the softest mode is the one with $q/2 = I$ and that this is a singular point of χ. This mode becomes unstable when $I = I_q$, where

$$\frac{1}{W} = 2\pi \ln \frac{2\Lambda}{I_q}. \qquad (13)$$

If we compare this equation to the gap equation in the absence of a field

$$\frac{1}{W} = 2\pi \ln \frac{2\Lambda}{\Delta_0} \qquad (14)$$

we see that

$$I_q = \Delta_0. \qquad (15)$$

Thus we have a window

$$\frac{\Delta_0}{\sqrt{2}} \lesssim I \lesssim \Delta_0 \qquad (16)$$

wherein neither the Fermi liquid, nor the homogeneous superconductor is the ground state. (Actually the lower limit should be lower since the BCS state has to 'beat' not the PFL but the inhomogeneous state. However, the condensation energy of the inhomogeneous state is quite small, and so is the change in the lower limit.)

If we put in the numbers we find that if B is in tesla and T_c in kelvin, the new phase should be stable in the interval

$$1.86T_c \lesssim B \lesssim 1.86\sqrt{2}T_c. \qquad (17)$$

We have used here the free-electron g factor. In practice g could be much higher, and the values of B correspondingly lower.
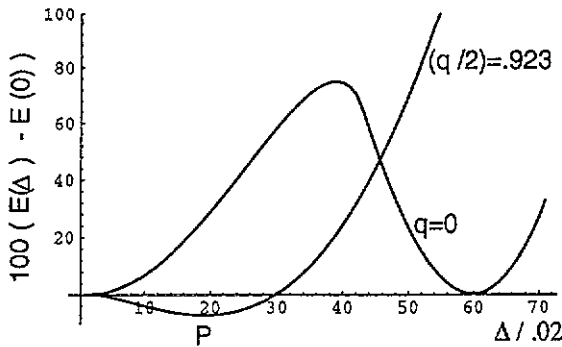

Figure 3. The energy of the the inhomogeneous and uniform BCS states relative to the polarized Fermi liquid (PFL) as a function of the order parameter Δ, just when the BCS energy (labelled $q = 0$) relative to that of the PFL vanishes. This is the point $I_F = \Delta_0/\sqrt{2}$ in figure 1. Note that that at this point the inhomogeneous state has a lower energy. The optimal momentum is somewhat larger than I. Given the energy scales, it clear that under a very slight reduction of I, the BCS state will dip below the inhomogeneous state. Thus we may take $I_F = \Delta_0/\sqrt{2}$ as the transition point between the uniform state and the inhomogeneous state.

Figure 3 shows a plot of the energy (in equation (7a)) versus Δ. The parameters $\Lambda = 100$, $I = 0.834$, $W = 0.031$ are chosen so as to bring the BCS energy (measured relative to the PFL) to zero, i.e., we have chosen $I = \Delta_0/\sqrt{2}$. Plotted on the same graph is the energy of the inhomogeneous state at $q/2 = 0.923$. Note that this $q/2$ is not equal to I. In other words, although $q/2 = I$ becomes unstable first, once the order sets in a larger $q/2$ does better and the graph shows the energy for the best $q/2$. If we lower I slightly, the BCS state will dip down very quickly to below the inhomogeneous state. Thus we may take $I = \Delta_0/\sqrt{2}$ to be essentially the field for the first-order transition to the inhomogeneous state. If I is raised further, the BCS graph will move upwards and so will the inhomogeneous state's energy at the minimum. As the minimum moves towards the origin, the optimal $q/2$ will move towards the current value of I. Finally for $I > \Delta_0$ the PFL will become stable to all pairing fluctuations. The transition to the PFL is clearly second order in mean-field theory.

From the energy formula it is easy to compute the specific heat by finding the density of states, and the magnetization by taking the I derivative. (Since $q/2$ and Δ are 'slaved' to I one may ask how they are to be handled during the derivation. In general there will be changes in E implicit and explicit to a change of I; however, at a point where $\partial E/\partial q$ and $\partial E/\partial \Delta$ vanish (as they do for us), only the explicit derivative matters.) The result is, in terms of the corresponding quantities for the PFL,

$$\frac{M}{M_{PFL}} = \frac{1}{2\pi I} \int_0^{2\pi} d\theta \sqrt{z^2 - \Delta^2} \left[\theta(z - \Delta) - \theta(-z - \Delta)\right] \tag{18}$$

$$\frac{C}{C_{PFL}} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{[z\theta(z - \Delta) - z\theta(-z - \Delta)]}{\sqrt{z^2 - \Delta^2}} \tag{19}$$

where Δ and $q/2$ are at their optimal values. Putting in the numbers we find that these ratios drop from the value of unity, at the FPL end, down to 0.92 and 0.79 respectively by the time we reach the situation shown in figure 3 (which is more or less the end of the inhomogeneous phase).

Let us next compute the exponent β which characterizes the onset of Δ. If we could write the energy near the transition as

$$E = -t\Delta^2 + u\Delta^4 + \cdots \tag{20}$$

where $t = \Delta_0 - I$, we would find as usual that $\Delta \simeq t^{1/2}$. If however, we analyse our energy function we find that

$$E = -t\Delta^2 + u\Delta^{5/2} + \cdots \tag{21}$$

leading to $\beta = 2$. In other words, *the energy function does not have an analytic expansion in Δ near the transition, which in turn can be traced to the fact that the optimal $q/2$ at the transition is a singular point of χ^{-1}. Thus even though we are doing mean-field theory, a nonclassical exponent emerges.*

We could go on and compute the electromagnetic response functions, but we do not, for the following reasons. Our analysis has shown that in the window

$$\frac{\Delta_0}{\sqrt{2}} \lesssim I \lesssim \Delta_0$$

an inhomogeneous state characterized by momentum of magnitude q (which is of the order I) has lower energy than either the PFL or the BCS state. However, our analysis does not choose a particular direction for q. Thus, as in problems involving, say, crystallization [6], all modes in this circle are degenerate. In this case the low-energy physics of the condensate is very much like that of a Fermi system, i.e., centred around a circle and not the origin. Recently Hohenberg and Swift [7] have shown how the shell elimination devised in [1] for fermions can be applied to this problem. For the present simply note that, instead of the plane wave we considered so far, a standing wave of the same wavelength might have a lower energy. The standard trick for deciding what exactly happens is to expand the energy functional in a power series in $\Delta(q)$, going out to fourth order. The fourth-order vertex (which is the particle–particle bubble with two more insertions in it) will determine which combination of modes is best near the transition. Unfortunately we cannot do this here since an analytical expansion in Δ does not exist. While we may expect that $\beta = 2$ may be insensitive to the actual form of the order parameter (plane wave versus cosine), features like specific heat will be very sensitive to the true ground state. For example if $\Delta \simeq \cos qr$, bands will be formed and the density of states can go up above the PFL value. The only thing that seems certain is that the specific heat will not be activated. Likewise the electromagnetic response and Meissner effect (for a small perpendicular field) will depend on the knowledge of the true ground state. This is a problem we have not solved. All we can say is that if the answer is a cosine, there will be an oscillation in spin density at the same wavelength.

We now consider the relation of our work to the Larkin–Ovchinnikov and Fulde–Ferrel [3, 4] results. These authors considered a three-dimensional superconductor with ferromagnetic impurities. The spins of these impurities were coupled to the electrons *via the exchange interaction* so that they could be represented on average by some external field that coupled only to spin. These authors too predict an inhomogeneous phase. We now list the differences between our work and theirs.

(i) These authors consider a system coupled to magnetic impurities while we need a clean system.

(ii) They evade the orbital complication of the magnetic field (even though they work in three dimensions) because the field really represents the exchange interaction with impurities. We evade it by considering a planar system with an external parallel field.

(iii) *Most importantly, they expect the inhomogeneous phase over a very narrow window* $0.707\Delta_0 \leqslant I \leqslant 0.755\Delta_0$ *in three dimensions whereas we expect it over a broad window* $0.707\Delta_0 \leqslant I \leqslant \Delta_0$ *in two dimensions, with* I *the controllable external field.* The window will be even wider in anisotropic systems (e.g. with elliptical FSs) because they are more one dimensional and will have roughly parallel segments of the FS over longer intervals.

(iv) Due to the fact that the optimal q is a singular value of χ^{-1} we find nonclassical exponents ($\beta = 2$) in the mean-field level, while they find classical exponents due to the fact that the optimal q, for small Δ, is $q = 1.2I$, which is a nonsingular point of the three-dimensional susceptibility. This allowed Larkin and Ovchinnikov to study mode coupling and to decide that the cosine order parameter is better than the plane wave. Till we can do a similar thing here it may be a reasonable to assume that the same thing happens in $d = 2$ also. In this case we may expect, as they do, that the spin density will will oscillate with period q and that the specific heat will be greater than for the PFL.

Consider next the work of Bulaevskii [5]. He looks at layers in a magnetic field and works out critical fields for any tilt and at finite temperature. His formula for zero tilt at zero temperature gives the window we quoted. The exponent $\beta = 2$ is reflected in the T dependence of his results. Like us, he does not consider spin–orbit coupling and for this reason could not explain some of the experiments [8] done at that time, as pointed out by Klemm, Luther and Beasley [9], who did include this effect. *It appears that the time is ripe to see this inhomogeneous phase. The conditions are that the material be as two dimensional as possible, clean, have low* Z *(to minimize spin–orbit effects from defects) and have a low* T_c *so that the required fields are not too large.* This will be interesting, not only in its own right, but also as a part of the family of systems discussed here. Of course, at finite T, there will be no ordered phase and a Kosterlitz–Thouless phase with algebraic order will take its place [10]. The phase structure will still be visible at small T and the $T = 0$ critical point will control the finite-T physics in the quantum critical region as in other problems [11, 12].

3. The spinless-fermion system with CDW order

We begin by reassuring our readers that what we mean by a spinless fermion can be experimentally realized in at least one way. In the idealized calculation we consider a system of spinless fermions that fills up a nested Fermi sea and ask what happens as we tamper with the nesting by changing the chemical potential. Assuming that all the action is centred around the immediate vicinity of the Fermi surface, i.e., that the interaction and the change in chemical potential I are small compared to the bandwidth, we can duplicate the same physics near the Fermi surface as follows. We take ordinary electrons and fill them up to, say, 45% (instead of half) of the Brillouin zone. Now we apply a magnetic field in the plane. This does nothing to the orbital motion but splits the spin-up and spin-down Fermi surfaces. For some choice of field, the bigger of the surfaces will begin to nest, while the other would have moved away in the opposite direction and will, we hope, not do anything interesting. The polarized electrons with a nested or nearly nested Fermi surface will be the spinless fermions of the theoretical calculation. Of course a CDW in this case will also imply a spin-density wave or SDW.

As in [1] we will consider free fermions with the dispersion relation which comes from nearest-neighbour hopping:

$$E(x, y) = -\cos x - r \cos y \tag{22}$$

where x and y stand for K_x and K_y. We choose anisotropic hopping since the generic physics we are talking about can be obscured by the van Hove singularities that lie on the Fermi surface when $r = 1$. We will work with $r > 1$. This energy relation satisfies the condition

$$E(x + \pi, y + \pi) = -E(x, y) \tag{23}$$

and the vector $Q = (\pi, \pi)$ connects points on the two branches of the Fermi surface, labelled by $\alpha = \pm 1$ as shown in figure 2. At half-filling the Fermi surface is defined by $E = 0$ or

$$y = \pm \, \cos^{-1} \left[-\frac{\cos x}{r} \right] \equiv \alpha \cos^{-1} \left[-\frac{\cos x}{r} \right]. \tag{24}$$

We will shortly use a variable ε that measures the energy from the FS of the free particles. In our problem, where the FS is at zero energy, $\varepsilon = E$.

Imagine turning on a chemical potential so that

$$E(x, y) = -\cos x - r \cos y + I. \tag{25}$$

The Fermi surface of the free fermions will now shrink to a smaller size and no longer nest. Of relevance is the combination

$$\frac{E(x + \pi + q, y + \pi + p) - E(x, y)}{2}$$

$$\equiv z(x, y, q, p)$$
$$= I - (q/2) \sin x - r(p/2) \sin y$$
$$= I - (q/2) \sin x - (p/2) \, \alpha \sqrt{r^2 - \cos^2 x} \equiv z_\alpha. \tag{26}$$

In the above formula, $\alpha = \pm 1$ is the branch index, and, as everywhere else in the paper, terms of quadratic order or higher in the small quantities I, p, q will be ignored.

Once again we limit ourselves to a band of energy Λ on either side of the free-electron FS. To this end we change variables from (x, y) to (x, ε), where ε measures the energy from the FS. (See [1] for details.) As a result

$$\int dx \, dy \to \int dx \, d\varepsilon \, J(r, x) \tag{27}$$

where the Jacobian is

$$J(r, x) = \frac{1}{\sqrt{r^2 - \cos^2 x}} \tag{28}$$

on the free-electron FS. We will ignore its deviation off the FS. It is useful to define

$$\bar{J} = \frac{1}{2\pi} \int_0^{2\pi} J(x) \, dx. \tag{29}$$

The mean-field Hamiltonian is

$$H = \frac{\Delta^2}{W} + \int_{-\Lambda}^{\Lambda} d\varepsilon \int_0^{2\pi} dx \, J(r, x) \left[c_+^\dagger(\varepsilon, x) c_+(\varepsilon, x) + c_-^\dagger(\varepsilon, x)) c_-(\varepsilon, x) \right] \varepsilon$$

$$+ \int_{-\Lambda}^{\Lambda} d\varepsilon \int_0^{2\pi} dx \, J(r, x) \Theta(|\varepsilon(x + \pi + q, y + \pi + p)| < \Lambda)$$

$$\times \left[\Delta c_+^\dagger(K) c_-(K + Q + iq + jp) + \Delta^* c_-^\dagger(K + Q + iq + jp) c_+(K) \right] \tag{30}$$

where now the subscripts $\pm$ on the operators refer to the branches of the FS.

To exhibit the intermediate phase, let us assemble the two ingredients: (i) the energy difference between the Fermi liquid and the (π, π) CDW state; and (ii) the inverse Fermi liquid susceptibility (in the RPA) to CDW formation at momentum $(x + \pi + q, y + \pi + p)$. The space between the zeros of these gives a window in I for the inhomogeneous state.

A standard mean-field calculation, as in the superconducting problem, shows that for $\Delta > I$, the energy difference between the (π, π) CDW state and Fermi liquid is

$$E(\Delta) - E(0) = \frac{\Delta^2}{W} + \int_0^{2\pi} dx \, J(r, x) \left[I^2 - \frac{\Delta^2}{2} - \Delta^2 \ln \frac{2\Delta}{\Delta} \right]. \tag{31}$$

This energy difference has a minimum at

$$\Delta_0 = 2\Lambda \exp\left[-\frac{1}{2\pi \bar{J} W} \right]. \tag{32}$$

Feeding this back into the energy formula we find that the I at which the (π, π) CDW state loses out to the Fermi liquid is

$$I_F = \frac{\Delta_0}{\sqrt{2}}. \tag{33}$$

(Note that once again the answer is consistent with the assumption $\Delta > I$ that went into its derivation.)

On summing the usual particle-hole bubbles, the inverse susceptibility of the Fermi liquid is found to be

$$\chi^{-1}(q, p) = \frac{1}{W} - \frac{1}{2} \sum_\alpha \int_0^{2\pi} dx \, J(r, x) \ln \frac{\Lambda}{|z_\alpha|} \tag{34}$$

where z_α is the value of z on branch α, as defined in equation (26). This gives

$$\chi^{-1}(q, p) = \frac{1}{W} - \int_0^{2\pi} dx \, J(r, x) \ln \frac{\Lambda}{\sqrt{(I - (q/2)\sin x)^2 - (p/2)^2(r^2 - \cos^2 x)}}. \tag{35}$$

First note that the uniform CDW (i.e., (π, π)) mode becomes singular when $I = I_0$ where

$$\frac{1}{W} = 2\pi \bar{J} \ln \frac{\Lambda}{I_0} \tag{36}$$

i.e., at

$$I_0 = \Lambda \exp\left[-\frac{1}{2\pi \bar{J} W} \right] = \frac{\Delta_0}{2}. \tag{37}$$

Next we consider just nonzero q. The instability occurs when

$$\frac{1}{W} = \int_0^{2\pi} J(r, x) \, dx \ln \frac{\Lambda}{|I - (q/2)\sin x|}. \tag{38}$$

Let us consider this equation at large r where the analysis is easier. Now $J \simeq 1/r$ and

$$\frac{1}{W} = \begin{cases} \dfrac{+2\pi}{r} \ln \dfrac{2\Lambda}{I + \sqrt{I^2 - (q/2)^2}} & q/2 < I \\[4mm] \dfrac{2\pi}{r} \ln \dfrac{2\Lambda}{q/2} & q/2 > I. \end{cases} \tag{39}$$

The optimal value is $q/2 = I$, and we find upon inverting the above that the I value at the onset of the instability is

$$I_q = 2\Lambda \exp\left[-\frac{r}{2\pi W} \right] = 2\Lambda \exp\left[-\frac{1}{2\pi \bar{J} W} \right] = \Delta_0. \tag{40}$$

At r not too large, but also not too close to unity (so that the singularity of the Jacobian does not dominate the integral), similar results hold.

So far we have found the same window as in the case of the superconductor, namely

$$\frac{\Delta_0}{\sqrt{2}} < I < \Delta_0. \tag{41}$$

But the window gets wider if we look at nonzero p. (This is also clear from figure 2.) Now we have

$$\frac{1}{W} = \int_0^{2\pi} J(r, x) \, dx \, \ln \frac{\Lambda}{\sqrt{I^2 - (p/2)^2(r^2 - \cos^2 x)}}$$

$$= \int_0^{2\pi} J(r, x) \, dx \, \ln \frac{\Lambda}{\sqrt{I^2 - (rp/2)^2 + (p^2/8) + (p^2/8)\cos 2x}}. \tag{42}$$

For large r, this integral is seen to be a maximum when $p = 2I/r$ where its value is $(2\pi/r) \ln(2\Lambda r/I_p)$ so

$$I_p = r\Delta_0. \tag{43}$$

The following table shows the deviation from this result for smaller r. In all these cases the optimal p was found to be $2I/r$. Note that the large-r results are not bad for r as low as 1.1.

r	1.1	1.5	2.5	3.5
$I_p/r\Delta_0$	0.83	0.93	0.98	0.99

The final phase diagram is as shown in figure 1(b). The main point is that as the anisotropy grows, so does the window, since the system is becoming increasingly one dimensional and the opposite faces of the Fermi surface are becoming globally parallel. (In the limit of parallel faces there exists a perfect nesting vector for any I.)

We will not consider the general case of a nonzero q and p since we have already seen a substantial window for observing the inhomogeneous phase. We also do not try to compute its energy as a function of Δ (as we succeeded in doing for the superconductor) because of the following problem. Recall that in the superconductor problem we coupled K to $-K + q$. That in turn must be coupled to $-(-K + q) + q = K$, which means only two modes are to be coupled. In the present case we coupled K to $K + (\pi, \pi) + q$, which in turn must be coupled to $K + 2q$ and so on. We did perform a calculation keeping just the first coupling to exhibit a variational wave function which could 'beat' the Fermi liquid and the (π, π) state inside the window.

Due to the fact that the optimal momentum p at the transition is a singular point of χ, we can once again expect nonclassical exponents. We guess that β will be larger than the classical value—probably equal to 2 once again.

3.1. Experimental realization

Let us now turn to the experimental realization of this system. We start with a system of real electrons not too close to half-filling, say at 45% filling. At zero temperature, this will probably be a BCS superconductor. Next we apply a parallel magnetic field. This is the problem we just studied. The BCS state will eventually pass via the inhomogeneous superconducting phase to the polarized Fermi liquid as the field 'rips apart' the Fermi surfaces of the up and own spins. (Here we must choose a material with a large g factor for the electron.) Focus on the two Fermi surfaces. One shrinks further away from half-filling and, we hope, does nothing interesting. The FS of the growing species will now come close

to nesting as the field is raised. *This species of polarized electrons constitutes the spinless fermions of the calculation, while the applied field, which adds a constant (Zeeman) energy per particle, will play the role of the changing chemical potential.* Note, however, that as the magnetic field grows and pushes the FS closer to nesting, the I in the spinless version *decreases*. At one point the system should enter the inhomogeneous CDW phase, which here means that the spin density also will oscillate with a momentum slightly different from (π, π). Eventually the system will jump by a first-order transition to the (π, π) state. In other words, when the chemical potential falls within a window, the system density locks at half-filling, taking particles from the reservoir, which in this case is the other species of spins. We should see a sudden increase in polarization which should then remain fixed as the field is increased since the up and down densities remain locked. (Unlike in the case of an ideal reservoir which can donate any number of particles at one energy, there will be a cost of energy when a macroscopic number of fermions have to be converted from the shrinking FS to the growing one. This will decrease the region of stability of the commensurate (π, π) CDW phase.) If the field is raised further, we are effectively changing the sign of I and so the same picture will appear in reverse: a first-order transition to the inhomogeneous state followed by a second-order transition to the Fermi liquid.

By design, the grand canonical picture applies to the above experiment, where the system of polarized electrons is indeed in contact with a reservoir (of opposite-spin electrons). The free parameter is the chemical potential or the applied magnetic field, and the particle density is chosen by the system to minimize its energy.
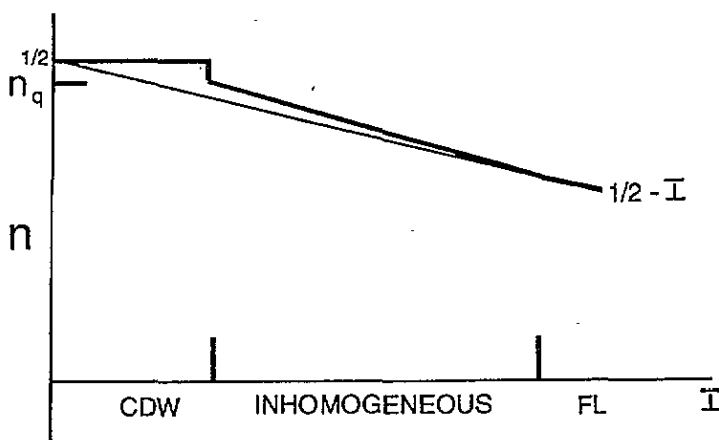

Figure 4. The phase structure as a function of number density n as well as chemical potential I. As I is raised, the system evolves along the x axis from the (π, π) CDW state to the inhomogeneous state to the PFL. As a function of n, the slightest doping from half-filling leads to phase coexistence until n comes down to n_q. Thereafter the behaviour is quite smooth.

Now the ideas discussed above also apply to other problems where the number density is the independent variable, say when a system of regular electrons at half-filling (as in a Hubbard model) is doped. Whereas if we take a commensurate system and change its chemical potential, it will initially ignore this and 'hang on' to its special density, here we want to forcibly change the number density and see what happens. We illustrate what is to be done in such a case by re-expressing the above spinless-fermion analysis in terms of number density. In other words we ask: suppose we took a half-filled system of spinless fermions and doped it, what would it do? Figure 4, which is schematic, helps us to analyse this. Along the x axis we vary I, the chemical potential, and cover the three phases. The

CDW phase refers to the (π, π) state. We find (in our calculation) that it has half-filling for all I up to the first-order transition. So we plot the corresponding n as a dark horizontal line going to the transition. A slight increase in I causes a first-order transition to the inhomogeneous state. It has a lower density n_q. Further increase in I causes a decline in n until we reach the second-order transition to the Fermi liquid. There is no jump in n. Hereafter n and I are related as in a free Fermi liquid: $n = \frac{1}{2} - I$. (The coefficient of I, the density of states, is set to unity). Now we can answer the question we posed. To find out what happens at any given density, choose a value for n, move horizontally to the dark line and come right down to the phase diagram on the I axis. Thus when $n = 1/2$ the system is in the (π, π) state. As n is lowered, ever so slightly, the system is stuck at the first-order transition and there will be coexistence of the (π, π) CDW and inhomogeneous states, the ratio being determined by n. When n is lowered down to n_q, the system will become all inhomogeneous. Thereafter we move smoothly along the dark line to the Fermi liquid transition and Fermi liquid phase. Thus, although a slight change in chemical potential does nothing to the (π, π) state, the slightest doping causes phase coexistence.

The above discussion has focused on breaking the nesting symmetry by changing the chemical potential. This is the choice with the closest analogy to the superconductor. We have also studied a case where nesting is destroyed by adding a second-neighbour term which causes additional wiggles on the FS without changing its volume. Here we find that the window for the inhomogeneous state shrinks. In the large-r limit, it goes to zero: the second-order transition at the origin in Δ space occurs when the first-order transition does. It is still interesting that the first-order transition far from the origin will be accompanied by huge fluctuations (at an incommensurate momentum) close to the origin. The reason for the uniform state being more stable in the presence of this kind of modification of the FS is clear: whereas changing I ruins nesting at (π, π) uniformly over the FS, the additional term due to second-neighbour hopping vanishes at some points on the old FS. Indeed any modification to the dispersion relation will necessarily be periodic and vanish somewhere, unless the periodic function is a constant, as when due to a change in chemical potential. Only in the case where the new free-electron FS does not intersect with the old FS anywhere do we have a big window for the inhomogeneous state.

A problem that we consider worth studying is that where both a second-neighbour term and chemical potential destroy the nested Fermi surface, with the latter as the control parameter.

4. The renormalization program

As mentioned earlier, the simple and intuitively appealing procedure of eliminating thin slices on either side of the FS works in the case of problems where the FS has special symmetries. The reason for this, as explained in detail in [1], is as follows. When we compute a one-loop graph for the β function, the two lines in the loop are either particle–hole (in the CDW case) or particle–particle (in the BCS case) lines. *To get a nonzero contribution to the flow, it is necessary that both lie in the thin shells being eliminated and also obey momentum conservation.* In the BCS case, if the overall momentum is zero, these lines have equal and opposite momenta. Given time-reversal symmetry, they have equal energies, so if one lies in one or the other shell (above or below the FS) that is being eliminated, so does the other, no matter what the direction of the line momentum is. In the CDW case, the momenta of the particle and hole lines differ by $Q = (\pi, \pi)$. Since this reverses the energy for the nested problem, again if one lies (anywhere) in the shell below or above the FS that is being eliminated, the other lies in the shell of opposite energy, also

being eliminated.

In both cases, no matter how many shells we eliminate, the couplings keep flowing, with each factor-of-s reduction in cut-off giving a contribution proportional to $\ln s$. The flow is off to the condensed state.

All this changes if we break the symmetry. Consider the CDW example. Now in order for one of the momenta to lie in a shell of thickness $d\Lambda$ and scatter into another one also being eliminated, its momentum must have a very specific *direction* that lies within a narrow range of size equal to the shell thickness divided by bandwidth. Thus in the shell elimination 'game', there will be no flow. This is not an insurmountable problem. In [1] another scheme called the field theory scheme (FTS) is invoked. Here one computes some physical quantity in the cut-off theory and sets the derivative with respect to the cut-off to zero to obtain the β function. We get the same flow as before for $\Lambda \ll I$. The flow does change character as Λ approaches I, as it should, and indicates a transition. But we are not very happy with this method since the FTS is not generally to be used when any of the other energies in the problem come close to Λ. This is because in the FTS one hopes that just the quartic interaction will be sufficient, and this requires that the cut-off dominate all other energies. In other words the smallness of ratios like p/Λ or q/Λ is what allows one to neglect higher operators. In the fully fledged Wilson–Kadanoff scheme there is, of course, no such restriction, *provided one works with operators of arbitrary complexity*, a prospect that we do not want to entertain. We are currently engaged in solving this problem.

Another approach is to follow Hertz [13]. Here one couples the fermions to a bosonic variable Δ as in a Hubbard–Stratonovich transformation, integrates the fermions out, and works with the effective action for Δ. As mentioned in [1], this is generally going to cause trouble since integrating out gapless degrees of freedom (anathema to the standard RG) can, and typically will, lead to a singular action for the remaining fields. If one did this for the usual BCS problem, one would find that the action cannot be expanded in powers of Δ. Even with the I term present, the action is nonpolynomial in Δ since the chosen momentum at the transition to the inhomogeneous phase is a singular point of the susceptibility. (In three dimensions this is not so because of phase space. In one dimension the singularity is even more pronounced.) Even if the action is polynomial, it may not have an analytic expansion in ω or q. Hertz has argued that in some cases we can still find the right scaling when this happens.

Returning to the general problem, one option is to keep both the fermions and bosons together and not eliminate only nonsingular modes. This is what one does in problems where gauge fields couple to fermions. In such cases the following question arises. For bosons the energy is measured from the origin in momentum space while for fermions it is measured from the FS. How are high- and low-energy modes to be defined? A boson that imparts a momentum parallel to the fermion momentum can take it beyond the cut-off (and is a high-energy boson) while the same boson, if it attaches itself to a fermion with a perpendicular momentum, will move along the FS and be a low-energy boson. A common solution is to pick a point on the fermion's FS and treat bosonic momenta in radial and angular directions distinctly. We find this approach to a rotationally invariant problem disturbing, even though, for reasons that we do not fully understand, it might work. On the other hand a recent paper by Altshuler, Ioffe and Millis [14] describes a class of problems where such a choice of coordinates *is* warranted. These authors consider a FS which is not rotationally invariant and of which only two small segments (parallel under translation by a vector Q) are important. In this case there *is* a preferred fermion direction, and using the fermion dispersion relation near these points to decide which bosons are high-energy ones and which are low-energy ones seems legitimate. This is one very promising possibility we

intend to fully digest when we resume our RG programme. However, for isotropic problems involving bosons and fermions, we believe that there is still need for improvement. In the meantime we can take the attitude that, since no problem is really isotropic, one may begin by attacking the gauge problems by starting with, say, an elliptical FS and trying the scheme of Altshuler *et al.*

5. Conclusion

We began with the result given in [1], that quantum phase transitions at zero coupling, which exist due to special symmetries of the FS, can be given an RG description in which shells of energies near the FS are symmetrically eliminated. Two examples given in [1] and discussed here were the planar superconductor (with a time-reversal-invariant spectrum) and planar spinless fermions at half-filling (with nesting symmetry of the FS). The goal was to extend the description to transitions at nonzero, but small, coupling so that a perturbative RG would be possible. To this end we had to come up with systems with such transitions, study them in the mean field to map out the phase structure, and then apply the RG to the transitions. The first step was accomplished by destroying the symmetries of the FS by adding a term I to the Hamiltonian. In the superconductor this was a parallel field and in the spinless-fermion case it was a chemical potential. The mean-field analysis shows us the following.

(i) The two systems have very similar phases and phase transitions.

(ii) The phase at large I (or small coupling) is a Fermi liquid. This is a polarized Fermi liquid in the BCS case. The next phase as I is reduced is an intermediate state with a condensate that has a nonzero momentum *with respect to the canonical value*, this being zero for the superconductor and (π, π) for the CDW. For this reason we call it the inhomogeneous phase. It has gapless excitations along with the nonzero order parameter. We have computed some of its properties for the superconductor. The last phase (at small I) is a state with condensate at the canonical momentum, which we call the homogeneous state even though $Q = (\pi, \pi)$ for the CDW.

(iii) The transition from the Fermi liquid to the inhomogeneous state is second order in the mean field but with nonclassical exponents: $\beta = 2$ for the superconductor. The nonclassical exponents arise because the chosen momentum for the condensate is a singular point of the Fermi liquid susceptibility. The transition from the intermediate state to the homogeneous state is first order.

(iv) The window for the inhomogeneous state is large due to two dimensionality. Indeed anisotropic systems have larger windows since they are more one dimensional. Recall that for the CDW case

$$\frac{\Delta_0}{\sqrt{2}} < I < r\Delta_0$$

where r is the anisotropy. We expect a similar enlargement of the window for anisotropic superconductors.

Our analysis has many limitations. First it is a mean-field analysis. The order of the transitions may change. (The analysis of Altshuler *et al* suggests both transitions may be first order.) However, given the large windows, the intermediate phases should not disappear due to fluctuations.

Our analysis ignores impurities and spin–orbit scattering. To see the phase discussed here will require clean, low-Z, two-dimensional systems and strong magnetic fields. For

the CDW case, where we try to move the FS using the field, a large g factor will help. It appears that we are at the threshold of having all of these.

Our analysis is at $T = 0$. This is not for technical reasons and can be readily overcome.

While our RG programme has naturally led us to study this *family* of systems, the systems themselves are not new. Rather than focus on individual systems, we have taken a slice through them and emphasized the common thread that runs through many of them. Our next step is to go through individual cases that have been studied experimentally and see whether or not they could belong to this family, i.e., to see whether they were clean enough, and two dimensional enough, to satisfy the various assumptions made in the derivation etc. For example, do our considerations possibly apply to granular superconductors [15, 16]? Do they apply to CDW transitions induced by application of pressure [17]? In the meantime we emphasize to the experimentalists that the intermediate inhomogeneous phase in any one system will not only be fascinating in its own right, but also in terms of the family.

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